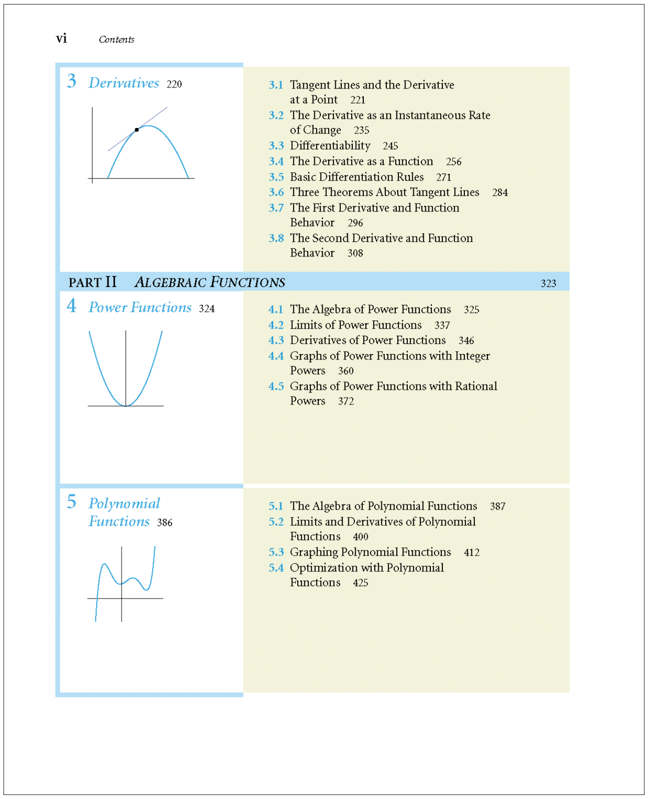
True Integration of Calculus, Algebra, and Precalculus

This text is **NOT** simply a cut-and-paste “just-in-time” combination of calculus and precalculus. Precalculus and algebra topics are woven **THROUGHOUT** the book and connected with the ideas of calculus. This provides a fresh perspective for students who are repeating these topics.

Inverted Structure with Functions as the Unifying Theme

Integrated Calculus is structured around the study of functions. Students learn basic theories and techniques of calculus and then use the knowledge to explore different types of functions. For example, after learning the basics of differentiation in Chapter 3, students apply what they know to increasingly complex functions in Chapters 4–12 as they investigate the algebra, precalculus, and calculus of each type of function. This reinforces skills through application to each new class of functions and is especially helpful to students with weak algebra skills.



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Algebra

Factoring and Solving Equations (Sections 0.2.2 and 0.2.3)

$a^2 - b^2 = (a-b)(a+b)$

$a^3 - b^3 = (a-b)(a^2+ab+b^2)$

$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots + a^2b^{n-3} + ab^{n-2} + b^{n-1})$

Quadratic Formula:

The solutions to $ax^2+bx+c=0$ are $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$.

$AB = 0 \iff A = 0$ or $B = 0$

$\frac{A}{B} = 0 \iff A = 0$ and $B \neq 0$

Rules of Fractions (Section 0.2.3)

$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$ $\quad \left(\frac{a}{b}\right)\left(\frac{c}{d}\right) = \frac{ac}{bd}$

$\frac{a}{b} - \frac{c}{d} = \frac{ad-bc}{bd}$ $\quad c\left(\frac{a}{b}\right) = \frac{ac}{b}$

$\frac{\left(\frac{a}{b}\right)}{\left(\frac{c}{d}\right)} = \left(\frac{a}{b}\right)\left(\frac{d}{c}\right) = \frac{ad}{bc}$ $\quad \frac{a}{\left(\frac{b}{c}\right)} = \frac{ac}{b}$ $\quad \frac{\left(\frac{a}{b}\right)}{c} = \frac{a}{bc}$

Inequalities (Sections 0.3.1–0.3.4)

If $a < b$ and $c > 0$, then $ac < bc$.
If $a < b$ and $c < 0$, then $ac > bc$.
If $a < b$, then for any c, $a + c < b + c$.
If $0 < a < b$, then $\frac{1}{a} > \frac{1}{b}$.
If A and B have the same sign, then $AB > 0$.
If A and B have opposite signs, then $AB < 0$.
$|x-c| < \delta \iff x \in (c-\delta, c+\delta)$
$|x-c| > \delta \iff x \in (-\infty, c-\delta) \cup (c+\delta, \infty)$
$x^2 < c^2 \iff x \in (-c, c)$.
$x^2 > c^2 \iff x \in (-\infty, -c) \cup (c, \infty)$.

Absolute Values (Sections 0.1.5 and 0.3.5)

$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$ $\quad |ab| = |a||b|$ $\quad \left|\frac{a}{b}\right| = \frac{|a|}{|b|}$

Triangle Inequality: $|a+b| \leq |a| + |b|$

Rules of Exponents (Sections 4.4.1–4.1.3 and 8.1.2)

$a^{-b} := \frac{1}{a^b}$ $\quad a^{\frac{1}{b}} := \sqrt[b]{a}$ $\quad a^0 := 1$ (for $a \neq 0$)

$c^a c^b = c^{a+b}$ $\quad (ab)^c = a^c b^c$ $\quad c^{ab} = (c^a)^b = (c^b)^a$

$\frac{c^a}{c^b} = c^{a-b}$ $\quad \left(\frac{a}{b}\right)^c = \frac{a^c}{b^c}$ $\quad c^{\frac{a}{b}} = \sqrt[b]{c^a} = (\sqrt[b]{c})^a$

Rules of Logarithms (Sections 9.1.1–9.1.4)

$\ln x := \log_e x$ $\quad \log x := \log_{10} x$

$\log_b(x^a) = a\log_b x$ $\quad \log_b(xy) = \log_b x + \log_b y$

$\log_b x = \frac{\log_a x}{\log_a b}$ $\quad \log_b(b^x) = x$

$\log_b 1 = 0$ $\quad \log_b(b) = 1$

$\log_b(\frac{1}{x}) = -\log_b x$ $\quad \log_b(\frac{x}{y}) = \log_b x - \log_b y$

$b^{\log_b x} = x$ $\quad \log_b x = y \iff b^y = x$

Trigonometric Identities (Sections 10.2.2 and 10.3.2–10.3.4)

$\csc x = \frac{1}{\sin x}$ $\quad \sec x = \frac{1}{\cos x}$

$\tan x = \frac{\sin x}{\cos x}$ $\quad \cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}$

Pythagorean: $\quad$ *Even/Odd:*

$\sin^2 x + \cos^2 x = 1$ $\quad \sin(-x) = -\sin x$

$\tan^2 x + 1 = \sec^2 x$ $\quad \cos(-x) = \cos x$

$1 + \cot^2 x = \csc^2 x$ $\quad \tan(-x) = -\tan x$

Sum:

$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

Difference:

$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \sin\beta\cos\alpha$

$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

Double-Angle: $\quad$ *Half-Angle:*

$\cos 2x = \cos^2 x - \sin^2 x$ $\quad \sin(\frac{x}{2}) = \pm\sqrt{\frac{1-\cos x}{2}}$

$\cos 2x = 1 - 2\sin^2 x$ $\quad \cos(\frac{x}{2}) = \pm\sqrt{\frac{1+\cos x}{2}}$

$\cos 2x = 2\cos^2 x - 1$ $\quad \sin^2 x = \frac{1-\cos 2x}{2}$

$\sin 2x = 2\sin x\cos x$ $\quad \cos^2 x = \frac{1+\cos 2x}{2}$

Shift:

$\cos(x - \frac{\pi}{2}) = \sin x$ $\quad \sin(x+2\pi) = \sin x$

$\sin(x + \frac{\pi}{2}) = \cos x$ $\quad \cos(x+2\pi) = \cos x$

Geometry

Distances (Sections 0.1.5 – 0.1.6)

- The *distance* between two real numbers a and b is $|b-a|$.
- *Distance Formula:* The distance between two points (x_1, y_1) and (x_2, y_2) in the plane is $\sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$.
- *Midpoint Formula:* The midpoint between two points (x_1, y_1) and (x_2, y_2) in the plane is the point $\left(\frac{x_1+x_2}{2}, \frac{x_1+x_2}{2}\right)$.

Triangles (Section 7.4.2)

- *Pythagorean Theorem:* If a right triangle has legs of length a and b, and a hypotenuse of length c, then $a^2 + b^2 = c^2$.
- *Law of Similar Triangles:* Suppose a right triangle has legs of length A and B, and hypotenuse of length C. For any right triangle similar to the first, with corresponding legs and hypotenuse of lengths a, b, and c, we have $\frac{a}{b} = \frac{A}{B}$, $\frac{b}{c} = \frac{B}{C}$, and $\frac{a}{c} = \frac{A}{C}$.

Lengths, Areas, and Volumes (Section 7.4.2)

- A *rectangle* with length l and width w has area $A = lw$ and perimeter $P = 2l + 2w$.
- A *triangle* with base b and height h has area $A = \frac{1}{2}bh$.
- A *circle* with radius r has area $A = \pi r^2$ and circumference $C = 2\pi r$. (See also Section 12.1.1)
- A *trapezoid* with base b and heights h_1 and h_2 has area $A = b\left(\frac{h_1+h_2}{2}\right)$. (See also Section 12.2.3)
- A *rectangular prism* with length l, width w, and height h has volume $V = lwh$ and surface area $SA = 2lw + 2wh + 2lh$.
- A *sphere* with radius r has volume $V = \frac{4}{3}\pi r^3$ and surface area $SA = 4\pi r^2$.
- A *right circular cylinder* with radius r and height h has volume $V = \pi r^2 h$, surface area $SA = 2\pi r h = 2\pi r^2$, and lateral surface area $LSA = 2\pi r h$.
- A *right circular cone* with raduis r and height h has volume $V = \frac{1}{3}\pi r^2 h$, surface area $SA = \pi r\sqrt{r^2+h^2} + \pi r^2$, and lateral surface area $LSA = \pi r\sqrt{r^2+h^2}$.
- A *shell* (*i.e.* a right circular cylinder with an interior cylinder of the same height removed) with small radii r_1, large radius r_2, and height h has volume $V = \pi(r_2^2 - r_1^2)h$. A shell with average radius $r = \frac{r_1+r_2}{2}$, thickness $\Delta x = r_2 - r_1$, and height h has volume $V = 2\pi r h \Delta x$. (See also Section 15.3.2)

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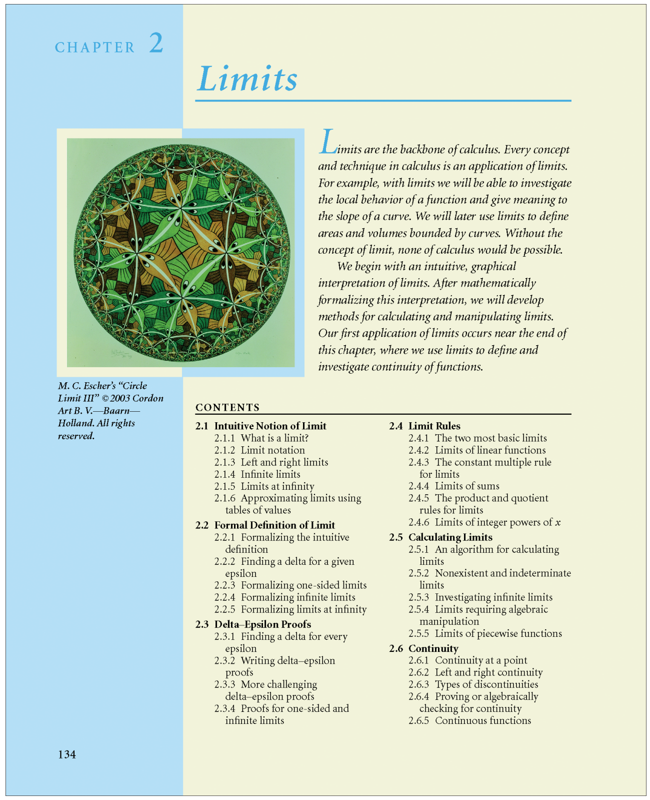
Handy Reference Pages

The Reference Pages provide a comprehensive summary of **ALL** symbols, formulas, theorems, and major topics that are used or developed throughout the text. Cross-references to appropriate sections of the book are cited to maximize effectiveness as a reference tool.

Chapter Openers

Each chapter opens with an example of mathematics in art or nature and a short introduction to the chapter. A detailed table of contents describes the sections and subsections of the chapter for easy reference.


CHAPTER 2

Limits

Limits are the backbone of calculus. Every concept and technique in calculus is an application of limits. For example, with limits we will be able to investigate the local behavior of a function and give meaning to the slope of a curve. We will later use limits to define areas and volumes bounded by curves. Without the concept of limit, none of calculus would be possible.

We begin with an intuitive, graphical interpretation of limits. After mathematically formalizing this interpretation, we will develop methods for calculating and manipulating limits. Our first application of limits occurs near the end of this chapter, where we use limits to define and investigate continuity of functions.

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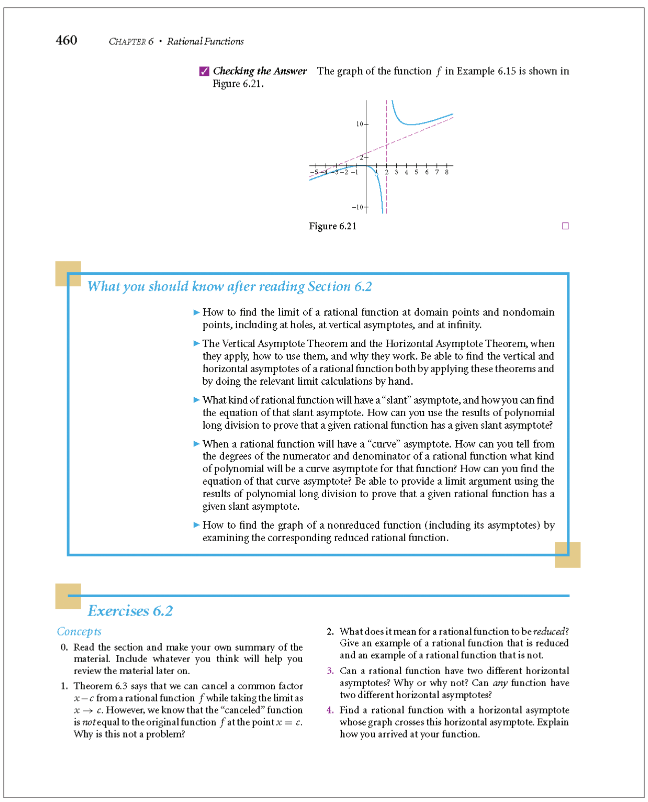

460 CHAPTER 6 • *Rational Functions*

☑ ***Checking the Answer*** The graph of the function f in Example 6.15 is shown in Figure 6.21.

Figure 6.21 □

What you should know after reading Section 6.2

- How to find the limit of a rational function at domain points and nondomain points, including at holes, at vertical asymptotes, and at infinity.
- The Vertical Asymptote Theorem and the Horizontal Asymptote Theorem, when they apply, how to use them, and why they work. Be able to find the vertical and horizontal asymptotes of a rational function both by applying these theorems and by doing the relevant limit calculations by hand.
- What kind of rational function will have a "slant" asymptote, and how you can find the equation of that slant asymptote. How can you use the results of polynomial long division to prove that a given rational function has a given slant asymptote?
- When a rational function will have a "curve" asymptote. How can you tell from the degrees of the numerator and denominator of a rational function what kind of polynomial will be a curve asymptote for that function? How can you find the equation of that curve asymptote? Be able to provide a limit argument using the results of polynomial long division to prove that a given rational function has a given slant asymptote.
- How to find the graph of a nonreduced function (including its asymptotes) by examining the corresponding reduced rational function.

Exercises 6.2

Concepts

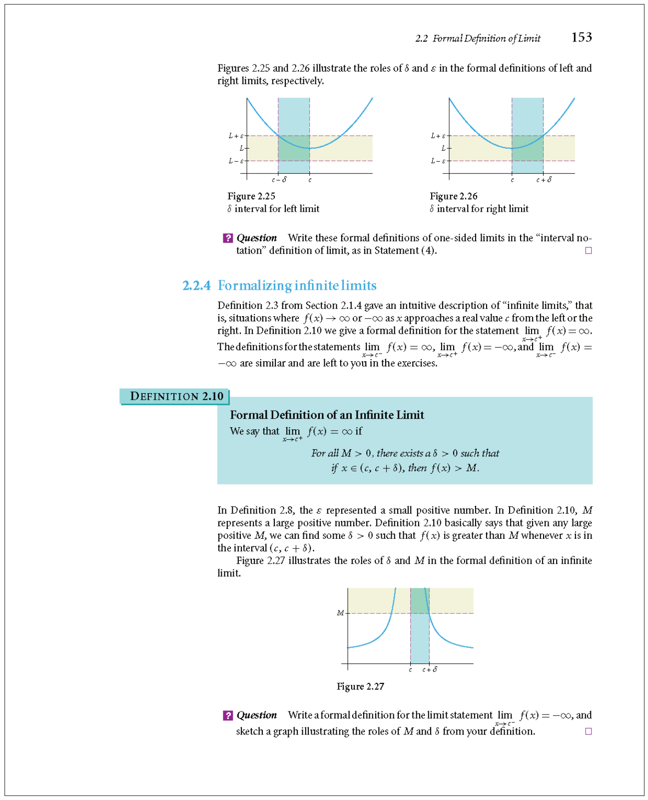
0. Read the section and make your own summary of the material. Include whatever you think will help you review the material later on.
1. Theorem 6.3 says that we can cancel a common factor $x-c$ from a rational function f while taking the limit as $x \to c$. However, we know that the "canceled" function is *not* equal to the original function f at the point $x = c$. Why is this not a problem?
2. What does it mean for a rational function to be *reduced*? Give an example of a rational function that is reduced and an example of a rational function that is not.
3. Can a rational function have two different horizontal asymptotes? Why or why not? Can *any* function have two different horizontal asymptotes?
4. Find a rational function with a horizontal asymptote whose graph crosses this horizontal asymptote. Explain how you arrived at your function.

Section Summaries

Each section ends with a *What you should know* summary that describes the main points of the section. This summary helps students test their understanding of the reading before going on to the homework exercises and helps them review for quizzes and tests.

Definitions

All important definitions are given titles and clearly set apart from the text for ease of reference. These definitions are worked into the exposition in the text and are often preceded by motivation and followed by clarification, examples, and illustrations.


2.2 Formal Definition of Limit 153

Figures 2.25 and 2.26 illustrate the roles of δ and ε in the formal definitions of left and right limits, respectively.

Figure 2.25
δ interval for left limit

Figure 2.26
δ interval for right limit

Question Write these formal definitions of one-sided limits in the "interval notation" definition of limit, as in Statement (4). □

2.2.4 Formalizing infinite limits

Definition 2.3 from Section 2.1.4 gave an intuitive description of "infinite limits," that is, situations where $f(x) \to \infty$ or $-\infty$ as x approaches a real value c from the left or the right. In Definition 2.10 we give a formal definition for the statement $\lim_{x \to c^+} f(x) = \infty$. The definitions for the statements $\lim_{x \to c^-} f(x) = \infty$, $\lim_{x \to c^+} f(x) = -\infty$, and $\lim_{x \to c^-} f(x) = -\infty$ are similar and are left to you in the exercises.

DEFINITION 2.10

Formal Definition of an Infinite Limit

We say that $\lim_{x \to c^+} f(x) = \infty$ if

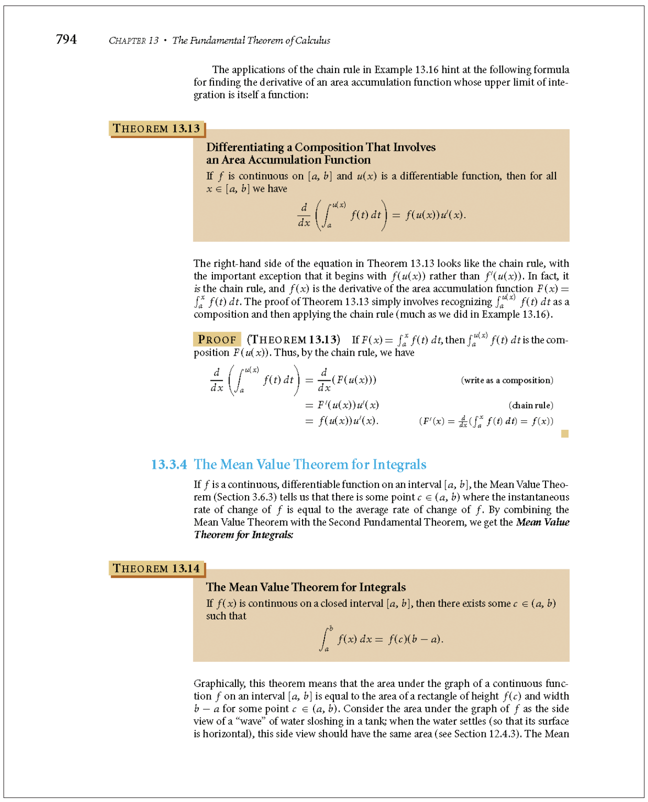
For all $M > 0$, there exists a $\delta > 0$ such that if $x \in (c, c + \delta)$, then $f(x) > M$.

In Definition 2.8, the ε represented a small positive number. In Definition 2.10, M represents a large positive number. Definition 2.10 basically says that given any large positive M, we can find some $\delta > 0$ such that $f(x)$ is greater than M whenever x is in the interval $(c, c + \delta)$.

Figure 2.27 illustrates the roles of δ and M in the formal definition of an infinite limit.

Figure 2.27

Question Write a formal definition for the limit statement $\lim_{x \to c^-} f(x) = -\infty$, and sketch a graph illustrating the roles of M and δ from your definition. □


794 CHAPTER 13 • The Fundamental Theorem of Calculus

The applications of the chain rule in Example 13.16 hint at the following formula for finding the derivative of an area accumulation function whose upper limit of integration is itself a function:

THEOREM 13.13

Differentiating a Composition That Involves an Area Accumulation Function

If f is continuous on $[a, b]$ and $u(x)$ is a differentiable function, then for all $x \in [a, b]$ we have

$$\frac{d}{dx}\left(\int_a^{u(x)} f(t)\,dt\right) = f(u(x))u'(x).$$

The right-hand side of the equation in Theorem 13.13 looks like the chain rule, with the important exception that it begins with $f(u(x))$ rather than $f'(u(x))$. In fact, it *is* the chain rule, and $f(x)$ is the derivative of the area accumulation function $F(x) = \int_a^x f(t)\,dt$. The proof of Theorem 13.13 simply involves recognizing $\int_a^{u(x)} f(t)\,dt$ as a composition and then applying the chain rule (much as we did in Example 13.16).

PROOF (THEOREM 13.13) If $F(x) = \int_a^x f(t)\,dt$, then $\int_a^{u(x)} f(t)\,dt$ is the composition $F(u(x))$. Thus, by the chain rule, we have

$$\begin{aligned} \frac{d}{dx}\left(\int_a^{u(x)} f(t)\,dt\right) &= \frac{d}{dx}(F(u(x))) && \text{(write as a composition)} \\ &= F'(u(x))u'(x) && \text{(chain rule)} \\ &= f(u(x))u'(x). && (F'(x) = \tfrac{d}{dx}(\textstyle\int_a^x f(t)\,dt) = f(x)) \end{aligned}$$

13.3.4 The Mean Value Theorem for Integrals

If f is a continuous, differentiable function on an interval $[a, b]$, the Mean Value Theorem (Section 3.6.3) tells us that there is some point $c \in (a, b)$ where the instantaneous rate of change of f is equal to the average rate of change of f. By combining the Mean Value Theorem with the Second Fundamental Theorem, we get the ***Mean Value Theorem for Integrals:***

THEOREM 13.14

The Mean Value Theorem for Integrals

If $f(x)$ is continuous on a closed interval $[a, b]$, then there exists some $c \in (a, b)$ such that

$$\int_a^b f(x)\,dx = f(c)(b - a).$$

Graphically, this theorem means that the area under the graph of a continuous function f on an interval $[a, b]$ is equal to the area of a rectangle of height $f(c)$ and width $b - a$ for some point $c \in (a, b)$. Consider the area under the graph of f as the side view of a "wave" of water sloshing in a tank; when the water settles (so that its surface is horizontal), this side view should have the same area (see Section 12.4.3). The Mean

Theorems

Theorems are also named and set apart from the text, with motivation and explanation in the text before and after each theorem. Proofs of theorems are written with the student in mind, with clear steps and discussion of the method of proof.

Examples

Numerous examples in the text illustrate important concepts and techniques and are named for easy reference. Students are encouraged to reflect on examples in the following ways:

- *Question* encourages students to be critical and active readers.
- *Caution* points out common pitfalls to avoid.
- *Checking the Answer* stresses the importance of verifying the correctness of answers independently and provides students with the skills to do so.

3.1 *Tangent Lines and the Derivative at a Point* 225

EXAMPLE 3.2 **Using the definition of derivative to find the exact slope of a tangent line**

Find the exact slope of the tangent line to the graph of $f(x) = x^2$ at $x = 3$.

Solution We use Definition 3.1 and find that the derivative of f at $x = 3$ is equal to

$$f'(3) = \lim_{h\to 0} \frac{f(3+h) - f(3)}{h} = \lim_{h\to 0} \frac{(3+h)^2 - (3)^2}{h}.$$

It remains only to calculate this limit. Note that if we "plug in" $h = 0$ at this point, we get the indeterminate form $\frac{0}{0}$, so we must do some algebra before evaluating this limit. We calculate

$$\begin{aligned} f'(3) &= \lim_{h\to 0} \frac{f(3+h) - f(3)}{h} = \lim_{h\to 0} \frac{(3+h)^2 - (3)^2}{h} \\ &= \lim_{h\to 0} \frac{(9+6h+h^2) - 9}{h} = \lim_{h\to 0} \frac{6h+h^2}{h} \\ &= \lim_{h\to 0} \frac{h(6+h)}{h} = \lim_{h\to 0}(6+h) = 6. \end{aligned}$$

Because the derivative of $f(x) = x^2$ at $x = 3$ is equal to 6, the line tangent to the graph of $y = x^2$ at the point (3, 9) has slope 6. □

Question Which limit rules did we use implicitly in Example 3.2? □

Caution Be careful that you do not "drop the limits" when doing a calculation like the one in Example 3.2. Since you are calculating a *limit* when you use the definition of derivative, each step of your calculation (until you actually "take the limit," of course) should include the expression "$\lim_{h\to 0}$." □

Checking the Answer To check whether the solution to Example 3.2 is reasonable, sketch the graph of f and see whether its tangent line at $x = 3$ appears to have a slope of 6. Figure 3.10 shows this graph. Note that the slope of the tangent line at $x = 3$ is positive, and considering the scale of the graph, a slope of 6 seems reasonable.

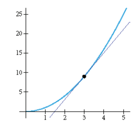


Figure 3.10

Alternatively, you could check the answer by using a graphing calculator to approximate the slope of the tangent line to $f(x) = x^2$ at $x = 3$. □

Caution You may have seen "rules" for differentiating in a previous course. Do *not* use these differentiation shortcuts at this point. Later, we will develop rules for quickly calculating derivatives, but at this point we must use the definition of derivative, as in Example 3.2. □

804 CHAPTER 14 • *Basic Integration Techniques*

ALGORITHM 14.1

Integration by Substitution

1. Choose some part of the integrand to be $u(x)$; usually a good choice for $u(x)$ is an "inside" function of a composition, whose derivative $u'(x)$ appears as a multiplicative factor of the integrand.
2. Differentiate the equation for $u(x)$ to find du; remember that $\frac{du}{dx} = u'(x)$, so $du = u'(x)\,dx$.
3. Change variables; that is, use the equations for $u(x)$ and du to rewrite the integrand, *including* the dx, *entirely* in terms of u and du.
4. Integrate the new, hopefully simpler integral.
5. Change variables back to x by substituting the formula for $u(x)$ for each u in the answer.

Algorithm 14.1 works for any integral in the form of Equation (1). The following example uses the algorithm of integration by substitution to find the integral from Example 14.1 and Example 14.2. (We keep repeating the same example so you can see how the substitution algorithm is derived from the chain rule.)

EXAMPLE 14.3 **Using integration by substitution**

Use integration by substitution to find $\int \cos x^2 (2x)\,dx$.

Solution Since x^2 is the "inside" of a composition, and its derivative $2x$ appears elsewhere in the integrand, we'll try setting $u = x^2$. Now we must differentiate $u(x)$ to find du:

$$\boxed{u = x^2} \implies \frac{du}{dx} = 2x \implies \boxed{du = 2x\,dx}.$$

By replacing x^2 by u (so $\cos(x^2) = \cos u$) and $2x\,dx$ by du (using the boxed equations), we have

$$\begin{aligned} \int \cos x^2(2x)\,dx &= \int \cos u\,du && \text{(since } u = x^2 \text{ and } du = 2x\,dx) \\ &= \sin u + C && \text{(integrate the new integral)} \\ &= \sin x^2 + C. && \text{(change variables back to } x) \end{aligned}$$ □

Caution In Example 14.3, the "new" integral $\int \cos u\,du$ was written *entirely* in terms of u and du. If there had been any x's left in the integral after we changed variables, it would have meant that we had made a bad choice for $u(x)$. We *cannot* integrate if the integrand involves both x and u. □

The method of integration by substitution works even if the integrand is not *exactly* in the form $f'(u(x))u'(x)$. The following example shows that we can use integration by substitution even when a constant multiple is "missing" from the integrand.

EXAMPLE 14.4 **Using integration by substitution when a constant multiple is "missing"**

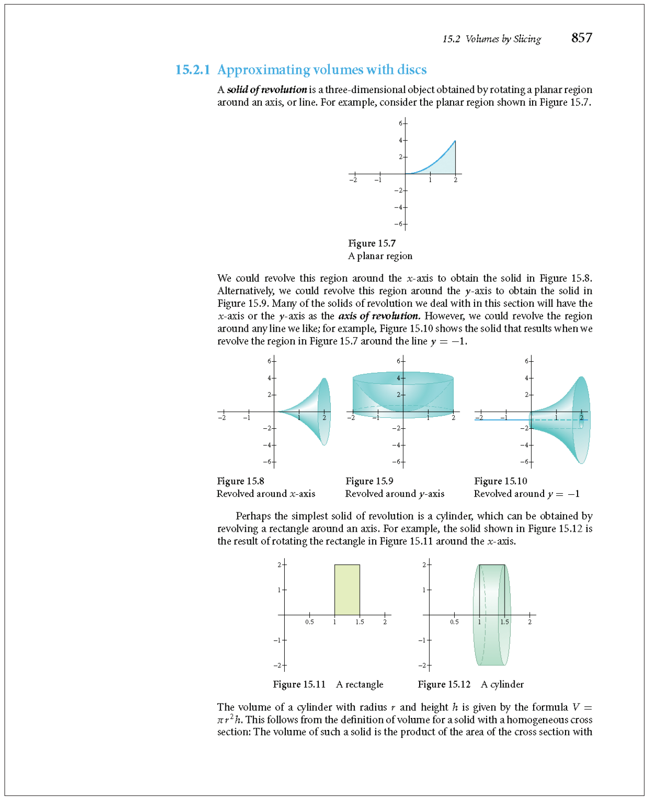
Use integration by substitution to find $\int x^2 e^{x^3+1}\,dx$.

Algorithms

While much of the book focuses on a conceptual and intuitive understanding of mathematics, important techniques are clearly outlined step-by-step and highlighted with detailed algorithms to enhance student understanding.

Readability

Integrated Calculus uses a linear, uncluttered presentation style that increases readability. Figures are included within the text itself, never banished to cluttered margins (where students may not look for them). The tone of the book is readable and conversational without the loss of an appropriate level of detail or mathematical precision.



15.2 *Volumes by Slicing* 857

15.2.1 Approximating volumes with discs

A ***solid of revolution*** is a three-dimensional object obtained by rotating a planar region around an axis, or line. For example, consider the planar region shown in Figure 15.7.

Figure 15.7
A planar region

We could revolve this region around the x-axis to obtain the solid in Figure 15.8. Alternatively, we could revolve this region around the y-axis to obtain the solid in Figure 15.9. Many of the solids of revolution we deal with in this section will have the x-axis or the y-axis as the ***axis of revolution.*** However, we could revolve the region around any line we like; for example, Figure 15.10 shows the solid that results when we revolve the region in Figure 15.7 around the line $y = -1$.

Figure 15.8
Revolved around x-axis

Figure 15.9
Revolved around y-axis

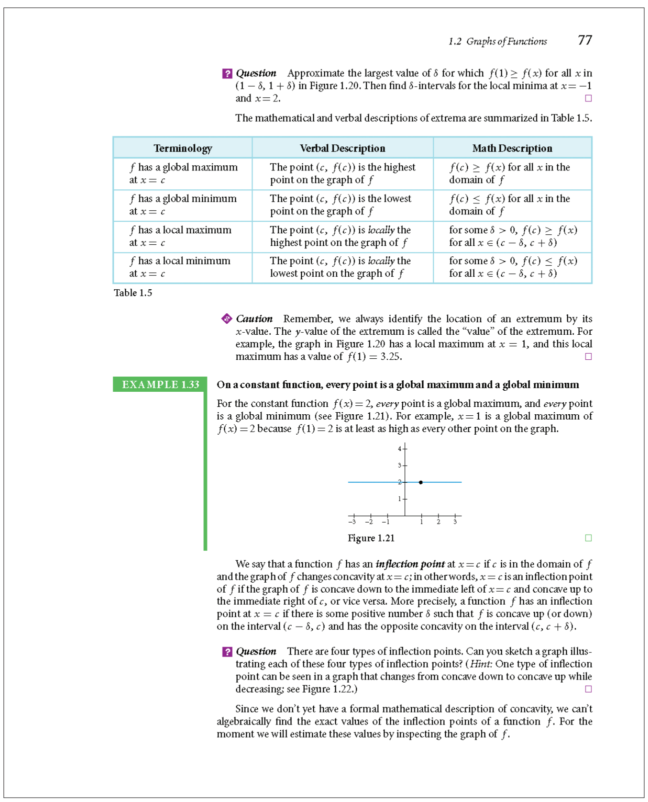
Figure 15.10
Revolved around $y = -1$

Perhaps the simplest solid of revolution is a cylinder, which can be obtained by revolving a rectangle around an axis. For example, the solid shown in Figure 15.12 is the result of rotating the rectangle in Figure 15.11 around the x-axis.

Figure 15.11 A rectangle

Figure 15.12 A cylinder

The volume of a cylinder with radius r and height h is given by the formula $V = \pi r^2 h$. This follows from the definition of volume for a solid with a homogeneous cross section: The volume of such a solid is the product of the area of the cross section with



1.2 *Graphs of Functions* 77

Question Approximate the largest value of δ for which $f(1) \geq f(x)$ for all x in $(1-\delta, 1+\delta)$ in Figure 1.20. Then find δ-intervals for the local minima at $x = -1$ and $x = 2$. □

The mathematical and verbal descriptions of extrema are summarized in Table 1.5.

Terminology	Verbal Description	Math Description
f has a global maximum at $x = c$	The point $(c, f(c))$ is the highest point on the graph of f	$f(c) \geq f(x)$ for all x in the domain of f
f has a global minimum at $x = c$	The point $(c, f(c))$ is the lowest point on the graph of f	$f(c) \leq f(x)$ for all x in the domain of f
f has a local maximum at $x = c$	The point $(c, f(c))$ is *locally* the highest point on the graph of f	for some $\delta > 0$, $f(c) \geq f(x)$ for all $x \in (c-\delta, c+\delta)$
f has a local minimum at $x = c$	The point $(c, f(c))$ is *locally* the lowest point on the graph of f	for some $\delta > 0$, $f(c) \leq f(x)$ for all $x \in (c-\delta, c+\delta)$

Table 1.5

Caution Remember, we always identify the location of an extremum by its x-value. The y-value of the extremum is called the "value" of the extremum. For example, the graph in Figure 1.20 has a local maximum at $x = 1$, and this local maximum has a value of $f(1) = 3.25$. □

EXAMPLE 1.33 **On a constant function, every point is a global maximum and a global minimum**

For the constant function $f(x) = 2$, *every* point is a global maximum, and *every* point is a global minimum (see Figure 1.21). For example, $x = 1$ is a global maximum of $f(x) = 2$ because $f(1) = 2$ is at least as high as every other point on the graph.

Figure 1.21 □

We say that a function f has an ***inflection point*** at $x = c$ if c is in the domain of f and the graph of f changes concavity at $x = c$; in other words, $x = c$ is an inflection point of f if the graph of f is concave down to the immediate left of $x = c$ and concave up to the immediate right of c, or vice versa. More precisely, a function f has an inflection point at $x = c$ if there is some positive number δ such that f is concave up (or down) on the interval $(c-\delta, c)$ and has the opposite concavity on the interval $(c, c+\delta)$.

Question There are four types of inflection points. Can you sketch a graph illustrating each of these four types of inflection points? (*Hint:* One type of inflection point can be seen in a graph that changes from concave down to concave up while decreasing; see Figure 1.22.) □

Since we don't yet have a formal mathematical description of concavity, we can't algebraically find the exact values of the inflection points of a function f. For the moment we will estimate these values by inspecting the graph of f.

Side-by-Side Intuitive and Formal Definitions

Intuitive and graphical approaches are presented side-by-side with the formal and rigorous mathematical definitions to complete students' understanding of terminology and concepts.

Exercises: Concepts and Skills

Each exercise set begins with “Problem 0” that stresses the importance of reading the text before attempting the exercises and helps students gather their thoughts before attempting problems. A great many “Concepts” problems in each section hold students accountable to the definitions and concepts from the reading and often include True/False problems and problems involving writing. “Skills” problems then provide practice and development for any calculational skills presented in the section.

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Exercises 10.4

Concepts

0. Read the section and make your own summary of the material. Include whatever you think will help you review the material later on.
1. Use a diagram of the unit circle to explain why $\sin\theta \approx \theta$ when θ is a small positive or negative angle.
2. Use a table of values to estimate $\lim_{\theta\to 0} \frac{\sin 2\theta}{\theta}$. Then check your approximation by examining a graph of $f(\theta) = \frac{\sin 2\theta}{\theta}$ near $\theta = 0$.
3. Use a table of values to estimate $\lim_{\theta\to 0} \frac{1-\cos 2\theta}{\theta}$. Then check your approximation by examining a graph of $f(\theta) = \frac{1-\cos 2\theta}{\theta}$ near $\theta = 0$.
4. What type of discontinuity does the function $f(x) = \frac{\sin x}{x}$ have at $x = 0$, and why?
5. Sketch a graph of $y = \sin x$ from memory. (*Hint:* Thinking about the unit circle may help you remember the features of the graph.) Using what you know about transformations of functions, use this graph to sketch the graphs of (a) $y = \sin x + 2$, (b) $y = \sin(x+2)$, (c) $y = 2\sin x$, and (d) $y = \sin 2x$.
6. Sketch a graph of $y = \tan x$ from memory. (*Hint:* Thinking about the unit circle may help you remember the features of the graph.) Using what you know about transformations of functions, use this graph to sketch the graphs of (a) $y = \tan x + 2$, (b) $y = \tan(x+2)$, (c) $y = 2\tan x$, and (d) $y = \tan 2x$.
7. Sketch a graph of $y = \sec x$ from memory. (*Hint:* Thinking about the unit circle may help you remember the features of the graph.) Using what you know about transformations of functions, use this graph to sketch the graphs of (a) $y = \sec x + 2$, (b) $y = \sec(x+2)$, (c) $y = 2\sec x$, and (d) $y = \sec 2x$.
8. List the periods of all of the six trigonometric functions.
9. Suppose you know that $\lim_{x\to 0} \sin x = 0$; explain why this means that $f(x) = \sin x$ is continuous at $x = 0$.
10. Suppose you know that $\lim_{x\to 0}(\cos x - 1) = 0$; explain why this means that $f(x) = \cos x$ is continuous at $x = 0$.
11. Explain why the limit $\lim_{x\to c} \cos x$ is equivalent to the limit $\lim_{h\to 0} \cos(c+h)$.
12. Suppose $x = c$ is a real number in the domain of $f(x) = \sec x$. How can we calculate $\lim_{x\to c} \sec x$, and why?
13. Guess the value of the limit $\lim_{x\to\infty} e^x \sin x$. Then check your answer by sketching a *good* graph. You should use both your calculator and your knowledge of the sine and exponential functions to sketch your graph.
14. Guess the value of the limit $\lim_{x\to\infty} \frac{\sin x}{e^x}$. Then check your answer by sketching a *good* graph. You should use both your calculator and your knowledge of the sine and exponential functions to sketch your graph.
15. Guess the value of the limit $\lim_{x\to\infty} \frac{e^x}{\sin x}$. Then check your answer by sketching a *good* graph. You should use both your calculator and your knowledge of the sine and exponential functions to sketch your graph.

■ Determine whether each of the following statements is true or false. If a statement is true, explain why. If a statement is false, explain why or provide a counterexample.

16. True or False: When θ is a small angle, the quantities $\sin\theta$ and θ are equal.
17. True or False: $y = \tan x$ is continuous at $x = 0$.
18. True or False: $y = \tan x$ is continuous at $x = \frac{\pi}{2}$.
19. True or False: The graph of $y = \csc x$ has vertical asymptotes at $x = k\pi$, for any integer k.
20. True or False: The graph of $y = \tan x$ intersects the x-axis at every integer multiple of π.
21. True or False: For any real number x, $\sin(x + 2\pi) = \sin x$.

Skills

■ Calculate the following limits (note that you will not be able to apply L’Hôpital’s Rule, since we do not yet know how to differentiate trigonometric functions). Show all work, and then check your answers with graphs.

22. $\lim_{x\to\pi} \frac{\cos x}{\sin x + 2}$
23. $\lim_{x\to -\frac{\pi}{4}} \frac{1}{\sin x + \cos x}$
24. $\lim_{x\to 0} \frac{1+\sin x}{1-\cos x}$
25. $\lim_{x\to 0} \frac{x}{\sin x}$
26. $\lim_{x\to 0} \frac{x}{1-\cos x}$
27. $\lim_{x\to 0} \frac{\sin 2x}{2x}$
28. $\lim_{x\to 0} \frac{\sin 3x}{5x}$
29. $\lim_{x\to 0} \frac{1-\cos 2x}{7x}$
30. $\lim_{x\to 0} \frac{3\sin x + x}{x}$
31. $\lim_{x\to 0} \frac{\sin^2 3x}{x^3 - x}$
32. $\lim_{x\to 0} \frac{\sin(3x^2)}{x^3 - x}$
33. $\lim_{x\to 0} \left(\frac{1}{2x} - \frac{2\cos 5x}{4x}\right)$
34. $\lim_{x\to 1^+} \frac{1-\cos\sqrt{x-1}}{\sqrt{x-1}}$
35. $\lim_{x\to 0^+} \frac{x^2\csc 3x}{1-\cos 2x}$
36. $\lim_{x\to\frac{\pi}{2}} \frac{\sin\left(x-\frac{\pi}{2}\right)}{\frac{\pi}{2}-x}$
37. $\lim_{x\to 0} \frac{x^2\cot x}{\sin x}$
38. $\lim_{x\to 0} \frac{\sec x\tan x}{x}$
39. $\lim_{x\to 0} 3x^2\cot^2 x$

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40. $\lim_{x\to 2} \frac{x^2-4}{\sin(x-2)}$
41. $\lim_{x\to 1} \frac{\sin^2(x-1)}{x^2-1}$
42. $\lim_{x\to 2} \frac{1-\cos(x-2)}{x^2+x-6}$
43. $\lim_{x\to\pi} \tan x$
44. $\lim_{x\to\frac{\pi}{2}} \tan x$
45. $\lim_{x\to\pi} \csc^2 x$
46. $\lim_{x\to\infty} \cos x$
47. $\lim_{x\to\infty} x\cos x$
48. $\lim_{x\to\infty} \frac{\cos x}{x}$
49. $\lim_{x\to 0} \sin\frac{1}{x}$
50. $\lim_{x\to 0} x\sin\frac{1}{x}$
51. $\lim_{x\to 0} x^2\sin\frac{1}{x}$

■ Use limits and the definition of continuity to determine whether each of the following functions is continuous at $x = 0$. Then use a graph to support your answer.

52. $f(x) = \begin{cases} \sin\left(\frac{1}{x}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$
53. $f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$
54. $f(x) = \begin{cases} x^2\sin\left(\frac{1}{x}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$

Applications

55. Consider a mass hanging from the ceiling at the end of a spring. If you pull down on the mass and let go, it will oscillate up and down according to the equation

$$s(t) = A\sin\left(\sqrt{\tfrac{k}{m}}\,t\right) + B\cos\left(\sqrt{\tfrac{k}{m}}\,t\right),$$

where $s(t)$ is the distance of the mass from its equilibrium position, m is the mass of the bob on the end of the spring, and k is a “spring coefficient” that measures how tight or stiff the spring is. The constants A and B depend on initial conditions—specifically, how far you pull down the mass (s_0) and the velocity at which you release the mass (v_0). This equation does *not* take into account any friction due to air resistance.

(a) Determine whether or not the limit of $s(t)$ as $t\to\infty$ exists. What does this say about the long-term behavior of the mass on the end of the spring?
(b) Explain how this limit is related to the fact that the equation above does *not* take friction due to air resistance into account.
(c) Suppose the bob at the end of the spring has a mass of 2 grams and that the coefficient for the spring is $k = 9$. Suppose also that the spring is released in such a way that $A = \sqrt{2}$ and $B = 2$. Use a calculator to graph the function $s(t)$ that describes the distance of the mass from its equilibrium position. Use your graph to support your answer to part (a).

56. In Problem 55 we gave an equation describing spring motion without air resistance. If we take into account friction due to air resistance, the mass will oscillate up and down according to the equation

$$s(t) = e^{\frac{-f}{2m}t}\left(A\sin\left(\frac{\sqrt{4km-f^2}}{2m}\,t\right) + B\cos\left(\frac{\sqrt{4km-f^2}}{2m}\,t\right)\right),$$

where m, k, A, and B are the constants described in Problem 55, and f is a positive “friction coefficient” that measures the amount of friction due to air resistance.

(a) Find the limit of $s(t)$ as $t\to\infty$. What does this say about the long-term behavior of the mass on the end of the spring?
(b) Explain how this limit is related to the fact that the equation above *does* take friction due to air resistance into account.
(c) Suppose the bob at the end of the spring has a mass of 2 grams, the coefficient for the spring is $k = 9$, and the friction coefficient is $f = 6$. Suppose also that the spring is released in such a way that $A = 4$ and $B = 2$. Use a calculator to graph the function $s(t)$ that describes the distance of the mass from its equilibrium position. Use your graph to support your answer to part (a).

Proofs

57. Use the fact that $\lim_{x\to 0} \frac{\sin x}{x} = 1$ to prove that $\lim_{x\to 0} \sin x = \sin 0$. Explain why this means that $f(x) = \sin x$ is continuous at $x = 0$.
58. Use the fact that $\lim_{x\to 0} \frac{1-\cos x}{x} = 0$ to prove that $\lim_{x\to 0}(\cos x - 1) = 0$. Explain why this means that $f(x) = \cos x$ is continuous at $x = 0$.
59. Use Theorem 10.12 and the sum identity for the sine function to prove that $\lim_{h\to 0} \sin(c+h) = \sin c$ for all real numbers c. Explain why this means that $f(x) = \sin x$ is continuous everywhere.
60. Use Theorem 10.12 and the sum identity for the cosine function to prove that $\lim_{h\to 0} \cos(c+h) = \cos c$ for all real numbers c. Explain why this means that $f(x) = \cos x$ is continuous everywhere.
61. Use the fact that the cosine function is continuous to prove that $f(x) = \sec x$ is continuous everywhere on its domain.
62. Use the fact that the sine and cosine functions are continuous to prove that $f(x) = \tan x$ is continuous everywhere on its domain.

Exercises: Applications

“Applications” problems link the material in the section to the real world. Some sections of the text are almost entirely devoted to applications. “Proofs” problems include many simple, easy proofs that students will be able to handle with confidence, as well as problems that ask students to repeat proofs from the reading.